Jaka suma?

Zadanie 1.

Wyznacz sumę: $S_n = 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n$.

Rozwiązanie

Zastosujemy wzór

$$S_n = \frac{q}{(1-q)^2}\left(1-(n+1)q^n + nq^{n+1}\right) = \frac{2}{(1-2)^2}\left(1-(n+1)2^n + n \cdot 2^{n+1}\right) =$$

$$= 2\left(1-(n+1)2^n + n \cdot 2^{n+1}\right) = 2\left(1-(n+1)2^n + 2 \cdot n \cdot 2^n\right) =$$

$$= 2\left(1-2^n n - 2^n + 2 \cdot n \cdot 2^n\right) = 2\big(1+2^n(n-1)\big) = 2 + 2^{n+1}(n-1)$$

Zadanie 2.

Wyznacz sumę: $T_n = 1 \cdot 3 + 3 \cdot 3^2 + 5 \cdot 3^3 + \cdots + (2n-1) \cdot 3^n$.

Rozwiązanie:

Naszą sumę możemy zapisać jako

$$T_n = \sum_{k=1}^{n}(2k-1)\cdot 3^k = \sum_{k=1}^{n}(3^k \cdot 2k - 3^k) = \sum_{k=1}^{n} 3^k \cdot 2k - \sum_{k=1}^{n} 3^k = 2\sum_{k=1}^{n} k \cdot 3^k - \sum_{k=1}^{n} 3^k$$

$$\sum_{k=1}^{n} k \cdot 3^k = \frac{3}{(1-3)^2}\left(1-(n+1)3^n + n3^{n+1}\right) = \frac{3}{4}\left(1 - 3^n n - 3^n + 3n \cdot 3^n\right) =$$

$$= \frac{3}{4}\left(1 + 2n \cdot 3^n - 3^n\right) = \frac{3}{4}\big(1 + 3^n(2n-1)\big)$$

$$2\sum_{k=1}^{n} k \cdot 3^k = \frac{3}{2}\big(1 + 3^n(2n-1)\big)$$

$$\sum_{k=1}^{n} 3^k = 3 \cdot \frac{1-3^n}{1-3} = 3 \cdot \frac{1-3^n}{-2} = -\frac{3}{2}(1-3^n)$$

$$T_n = 2\sum_{k=1}^{n} k \cdot 3^k - \sum_{k=1}^{n} 3^k = \frac{3}{2}\big(1+3^n(2n-1)\big) + \frac{3}{2}(1-3^n) =$$

$$= \frac{3}{2} \cdot \Big(\big(1+3^n(2n-1)\big) + (1-3^n)\Big) = \frac{3}{2}\big(2 + 3^n(2n-2)\big) = 3 + 3^{n+1}(n-1)$$